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Research Article

On Fixed Point Theorems of Mixed Monotone Operators

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We obtain some new existence and uniqueness theorems of positive fixed point of mixed monotone operators in Banach spaces partially ordered by a cone. Some results are new even for increasing or decreasing operators.

1. Introduction

Mixed monotone operators were introduced by Guo and Lakshmikantham in [1] in 1987. Thereafter many authors have investigated these kinds of operators in Banach spaces and obtained a lot of interesting and important results. They are used extensively in nonlinear differential and integral equations. In this paper, we obtain some new existence and uniqueness theorems of positive fixed point of mixed monotone operators in Banach spaces partially ordered by a cone. Some results are new even for increasing or decreasing operators.

Let the real Banach space E be partially ordered by a cone P of E, that is, $x \le y$ if and only if $y - x \in P$. $A : P \times P \to P$ is said to be a mixed monotone operator if $A(x, y)$ is increasing in x and decreasing in y, that is, u_i, v_i $(i = 1, 2) \in P, u_1 \le u_2, v_1 \ge v_2$ implies $A(u_1, v_1) \le A(u_2, v_2)$. Element $x \in P$ is called a fixed point of A if $A(x, x) = x$.

Recall that cone P is said to be solid if the interior $\mathring{P}$ is nonempty, and we denote $x \gg 0$ if $x \in \mathring{P}$. P is normal if there exists a positive constant N such that $0 \le x \le y$ implies $\|x\| \le N\|y\|$, N is called the normal constant of P.

For all $x, y \in E$, the notation $x \sim y$ means that there exist $\lambda > 0$ and $\mu > 0$ such that $\lambda x \le y \le \mu x$. Clearly, $\sim$ is an equivalence relation. Given $h > \theta$ (i.e., $h \ge \theta$ and $h \ne \theta$), we denote by P_h the set $P_h = \{x \in E \mid x \sim h\}$. It is easy to see that $P_h \subset P$ is convex and $\lambda P_h = P_h$ for all $\lambda > 0$. If $\overset{\circ}{P} \neq \emptyset$ and $h \in \overset{\circ}{P}$, it is clear that $P_h = \overset{\circ}{P}$.

All the concepts discussed above can be found in [2, 3]. For more facts about mixed monotone operators and other related concepts, the reader could refer to [4–9] and some of the references therein.

2. Main Results

In this section, we present our main results. To begin with, we give the definition of τ-φ-concave-convex operators.

Definition 2.1. Let E be a real Banach space and P a cone in E. We say an operator $A : P \times P \to P$ is τ-φ-concave-convex operator if there exist two positive-valued functions $\tau(t), \varphi(t)$ on interval (a, b) such that

(H_1) $\tau(t) : (a, b) \to (0, 1)$ is a surjection,

(H_2) $\varphi(t) > \tau(t)$, for all $t \in (a, b)$,

(H_3) $A(\tau(t)x, (1/\tau(t))y) \geq \varphi(t)A(x, y)$, for all $t \in (a, b)$, $x, y \in P$.

Theorem 2.2. *Let P be normal cone of E, and let $A : P \times P \to P$ be a mixed monotone and τ-φ-concave-convex operator. In addition, suppose that there exists $h > \theta$ such that $A(h, h) \in P_h$, then A has exactly one fixed point x^* in P_h. Moreover, constructing successively the sequence*

$$x_n = A(x_{n-1}, y_{n-1}), \qquad y_n = A(y_{n-1}, x_{n-1}), \quad n = 1, 2, \ldots, \tag{2.1}$$

for any initial $x_0, y_0 \in P_h$, one has

$$\|x_n - x^*\| \longrightarrow 0, \qquad \|y_n - x^*\| \longrightarrow 0, \quad n \longrightarrow \infty. \tag{2.2}$$

Proof. We divide the proof into 3 steps.

Step 1. We prove that A has a fixed point in P_h.

Since $A(h, h) \in P_h$, we can choose a sufficiently small number $e_0 \in (0, 1)$ such that

$$e_0 h \leq A(h, h) \leq \frac{1}{e_0} h. \tag{2.3}$$

It follows from (H_1) that there exists $t_0 \in (a, b)$ such that $\tau(t_0) = e_0$, and hence

$$\tau(t_0) h \leq A(h, h) \leq \frac{1}{\tau(t_0)} h. \tag{2.4}$$

By (H_2), we know that $\varphi(t_0)/\tau(t_0) > 1$. So, we can take a positive integer k such that

$$\left(\frac{\varphi(t_0)}{\tau(t_0)}\right)^k \geq \frac{1}{\tau(t_0)}. \tag{2.5}$$

It is clear that

$$\left(\frac{\tau(t_0)}{\varphi(t_0)}\right)^k \leq \tau(t_0). \tag{2.6}$$

Let $u_0 = [\tau(t_0)]^k h, v_0 = (1/[\tau(t_0)]^k)h$. Evidently, $u_0, v_0 \in P_h$ and $u_0 = [\tau(t_0)]^{2k} v_0 < v_0$. By the mixed monotonicity of A, we have $A(u_0, v_0) \leq A(v_0, u_0)$. Further, combining the condition (H_3) with (2.4) and (2.6), we have

$$\begin{aligned} A(u_0, v_0) &= A\left([\tau(t_0)]^k h, \frac{1}{[\tau(t_0)]^k} h \right) \\ &= A\left(\tau(t_0)[\tau(t_0)]^{k-1} h, \frac{1}{\tau(t_0)} \frac{1}{[\tau(t_0)]^{k-1}} h \right) \\ &\geq \varphi(t_0) A\left([\tau(t_0)]^{k-1} h, \frac{1}{[\tau(t_0)]^{k-1}} h \right) \geq \cdots \\ &\geq (\varphi(t_0))^k A(h,h) \geq (\varphi(t_0))^k \tau(t_0) h \\ &\geq [\tau(t_0)]^k h = u_0. \end{aligned} \tag{2.7}$$

For $t \in (a,b)$, from (H_3), we get

$$A(x,y) = A\left(\tau(t)\frac{1}{\tau(t)}x, \frac{1}{\tau(t)}\tau(t)y\right) \geq \varphi(t) A\left(\frac{1}{\tau(t)}x, \tau(t)y\right), \tag{2.8}$$

and hence

$$A\left(\frac{1}{\tau(t)}x, \tau(t)y\right) \leq \frac{1}{\varphi(t)} A(x,y), \quad \forall t \in (a,b),\ x,y \in P. \tag{2.9}$$

Thus, we have

$$\begin{aligned}
A(v_0,u_0) &= A\left(\frac{1}{[\tau(t_0)]^k}h,[\tau(t_0)]^k h\right)\\
&= A\left(\frac{1}{\tau(t_0)}\frac{1}{[\tau(t_0)]^{k-1}}h,\tau(t_0)[\tau(t_0)]^{k-1}h\right)\\
&\le \frac{1}{\varphi(t_0)}A\left(\frac{1}{[\tau(t_0)]^{k-1}}h,[\tau(t_0)]^{k-1}h\right)\le\cdots\\
&\le \frac{1}{(\varphi(t_0))^k}A(h,h)\le\frac{1}{(\varphi(t_0))^k}\cdot\frac{1}{\tau(t_0)}h\\
&\le \frac{1}{[\tau(t_0)]^k}h = v_0.
\end{aligned}\tag{2.10}$$

Construct successively the sequences

$$u_n = A(u_{n-1},v_{n-1}),\qquad v_n = A(v_{n-1},u_{n-1}),\quad n=1,2,\ldots.\tag{2.11}$$

It follows from (2.7), (2.10), and the mixed monotonicity of A that

$$u_0\le u_1\le\cdots\le u_n\le\cdots\le v_n\le\cdots\le v_1\le v_0.\tag{2.12}$$

Note that $u_0=[\tau(t_0)]^{2k}v_0$, so we can get $u_n\ge u_0\ge[\tau(t_0)]^{2k}v_0\ge[\tau(t_0)]^{2k}v_n$, $n=1,2,\ldots$. Let

$$r_n=\sup\{r>0\mid u_n\ge rv_n\},\quad n=1,2,\ldots.\tag{2.13}$$

Thus, we have $u_n\ge r_nv_n$, $n=1,2,\ldots$, and then

$$u_{n+1}\ge u_n\ge r_nv_n\ge r_nv_{n+1},\quad n=1,2,\ldots.\tag{2.14}$$

Therefore, $r_{n+1}\ge r_n$, that is, $\{r_n\}$ is increasing with $\{r_n\}\subset(0,1]$. Suppose that $r_n\to r^*$ as $n\to\infty$, then $r^*=1$. Indeed, suppose to the contrary that $0<r^*<1$. By (H_1), there exists $t_1\in(a,b)$ such that $\tau(t_1)=r^*$. We distinguish two cases.

Case 1. There exists an integer N such that $R_N=r^*$. In this case, we know that $r_n=r^*$ for all $n\ge N$. So, for $n\ge N$, we have

$$u_{n+1}=A(u_n,v_n)\ge A\left(r_nv_n,\frac{1}{r_n}u_n\right)=A\left(\tau(t_1)v_n,\frac{1}{\tau(t_1)}u_n\right)\ge\varphi(t_1)v_{n+1}.\tag{2.15}$$

By the definition of r_n, we get $r_{n+1}=r^*\ge\varphi(t_1)>\tau(t_1)=r^*$, which is a contradiction.

Case 2. If for all integer n, $r_n < r^*$, then $0 < r_n/r^* < 1$. By (H_1), there exists $s_n \in (a,b)$ such that $\tau(s_n) = r_n/r^*$. So, we have

$$\begin{aligned} u_{n+1} &= A(u_n, v_n) \ge A\left(r_n v_n, \frac{1}{r_n} u_n\right) = A\left(\frac{r_n}{r^*} r^* v_n, \frac{r^*}{r_n}\frac{1}{r^*} u_n\right) = A\left(\tau(s_n) r^* v_n, \frac{1}{\tau(s_n)}\frac{1}{r^*} u_n\right) \\ &\ge \varphi(s_n) A\left(r^* v_n, \frac{1}{r^*} u_n\right) \ge \varphi(s_n)\varphi(t_1) v_{n+1}. \end{aligned} \tag{2.16}$$

By the definition of r_n, we have

$$r_{n+1} \ge \varphi(s_n)\varphi(t_1) \ge \tau(s_n)\varphi(t_1) = \frac{r_n}{r^*}\varphi(t_1). \tag{2.17}$$

Let $n \to \infty$, we get $r^* \ge \varphi(t_1) > \tau(t_1) = r^*$, which is also a contradiction. Thus, $\lim_{n\to\infty} r_n = 1$. For any natural number p, we have

$$\begin{aligned} \theta &\le u_{n+p} - u_n \le v_n - u_n \le (1 - r_n) v_0, \\ \theta &\le v_{n+p} - v_n \le v_n - u_n \le (1 - r_n) v_0. \end{aligned} \tag{2.18}$$

Since P is normal, we have

$$\begin{aligned} \left\|u_{n+p} - u_n\right\| &\le N(1 - r_n)\|v_0\| \quad (\text{as } n \to \infty), \\ \left\|v_{n+p} - v_n\right\| &\le N(1 - r_n)\|v_0\| \quad (\text{as } n \to \infty). \end{aligned} \tag{2.19}$$

Here, N is the normality constant.

So, $\{u_n\}$ and $\{v_n\}$ are Cauchy sequences. Because E is complete, there exist u^*, v^* such that $u_n \to u^*, v_n \to v^*$ $(n \to \infty)$. By (2.12), we know that $u_n \le u^* \le v^* \le v_n$ and

$$\theta \le v^* - u^* \le (1 - r_n) v_0. \tag{2.20}$$

Further,

$$\|v^* - u^*\| \le N(1 - r_n)\|v_0\| \quad (n \to \infty), \tag{2.21}$$

and thus $u^* = v^*$. Let $x^* := u^* = v^*$, we obtain

$$u_{n+1} = A(u_n, v_n) \le A(x^*, x^*) \le A(v_n, u_n) = v_{n+1}. \tag{2.22}$$

Let $n \to \infty$, we get $x^* = A(x^*, x^*)$. That is, x^* is a fixed point of A in P_h.

Step 2. We prove that x^* is the unique fixed point of A in P_h.

In fact, suppose that $\overline{x}$ is a fixed point of A in P_h. Since $x^*, \overline{x} \in P_h$, there exist positive numbers $\beta > \alpha > 0$ such that $\alpha\overline{x} \leq x^* \leq \beta\overline{x}$. Let $e_1 = \sup\{e > 0 \mid (1/e)\overline{x} \geq x^* \geq e\overline{x}\}$. Evidently, $e_1 \in (0,1]$. We now prove that $e_1 = 1$. If otherwise, $0 < e_1 < 1$. From (H_1), there exists $t_2 \in (a,b)$ such that $\tau(t_2) = e_1$. Then,

$$\begin{aligned} x^* &= A(x^*, x^*) \geq A\left(e_1\overline{x}, \frac{1}{e_1}\overline{x}\right) = A\left(\tau(t_2)\overline{x}, \frac{1}{\tau(t_2)}\overline{x}\right) \\ &\geq \varphi(t_2) A(\overline{x}, \overline{x}) = \varphi(t_2)\overline{x}. \\ x^* &= A(x^*, x^*) \leq A\left(\frac{1}{e_1}\overline{x}, e_1\overline{x}\right) = A\left(\frac{1}{\tau(t_2)}\overline{x}, \tau(t_2)\overline{x}\right) \\ &\leq \frac{1}{\varphi(t_2)} A(\overline{x}, \overline{x}) = \frac{1}{\varphi(t_2)}\overline{x}. \end{aligned} \tag{2.23}$$

Since $\varphi(t_2) > \tau(t_2) = e_1$, this contradicts the definition of e_1. Hence, $e_1 = 1$, thus, $x^* = \overline{x}$. Therefore, A has a unique fixed point x^* in P_h.

Step 3. We prove (2.2).

For any $x_0, y_0 \in P_h$, we can choose a small number $e_2 \in (0,1)$ such that

$$e_2 h \leq x_0 \leq \frac{1}{e_2}h, \qquad e_2 h \leq y_0 \leq \frac{1}{e_2}h. \tag{2.24}$$

Also from (H_1), there is $t_3 \in (a,b)$ such that $\tau(t_3) = e_2$, and hence

$$\tau(t_3)h \leq x_0 \leq \frac{1}{\tau(t_3)}h, \qquad \tau(t_3)h \leq y_0 \leq \frac{1}{\tau(t_3)}h. \tag{2.25}$$

We can choose a sufficiently large integer m, such that

$$\left[\frac{\varphi(t_3)}{\tau(t_3)}\right]^m \geq \frac{1}{\tau(t_3)}. \tag{2.26}$$

Let $\overline{u_0} = [\tau(t_3)]^m h, \overline{v_0} = (1/[\tau(t_3)]^m)h$. It is easy to see that $\overline{u_0}, \overline{v_0} \in P_h$ and $\overline{u_0} \leq x_0 \leq \overline{v_0}, \overline{u_0} \leq y_0 \leq \overline{v_0}$. Put $\overline{u_n} = A(\overline{u_{n-1}}, \overline{v_{n-1}})$, $\overline{v_n} = A(\overline{v_{n-1}}, \overline{u_{n-1}})$, $x_n = A(x_{n-1}, y_{n-1})$, $y_n = A(y_{n-1}, x_{n-1})$, $n = 1, 2, \ldots$. Similarly to Step 1, it follows that there exists $y^* \in P_h$ such that $A(y^*, y^*) = y^*$, $\lim_{n\to\infty}\overline{u_n} = \lim_{n\to\infty}\overline{v_n} = y^*$. By the uniqueness of fixed points of operator A in P_h, we get $y^* = x^*$. And by induction, $\overline{u_n} \leq x_n \leq \overline{v_n}, \overline{u_n} \leq y_n \leq \overline{v_n}$, $n = 1, 2 \ldots$. Since P is normal, we have $\lim_{n\to\infty} x_n = x^*$, $\lim_{n\to\infty} y_n = x^*$. □

3. Concerned Remarks and Corollaries

If we suppose that the operator $A : P_h \times P_h \to P_h$ or $A : \mathring{P} \times \mathring{P} \to \mathring{P}$ with P is a solid cone, then $A(h,h) \in P_h$ is automatically satisfied. This proves the following corollaries.

Corollary 3.1. *Let P be a normal cone of E, and let $A : P_h \times P_h \longrightarrow P_h$ be a mixed monotone and τ-φ-concave-convex operator, then A has exactly one fixed point x^* in P_h. Moreover, constructing successively the sequence*

$$x_n = A(x_{n-1}, y_{n-1}), \qquad y_n = A(y_{n-1}, x_{n-1}), \quad n = 1, 2, \ldots, \tag{3.1}$$

for any initial $x_0, y_0 \in P_h$, one has

$$\|x_n - x^*\| \longrightarrow 0, \qquad \|y_n - x^*\| \longrightarrow 0, \quad n \longrightarrow \infty. \tag{3.2}$$

Corollary 3.2. *Let P be a normal solid cone of E, and let $A : \overset{\circ}{P} \times \overset{\circ}{P} \longrightarrow \overset{\circ}{P}$ be a mixed monotone and τ-φ-concave-convex operator, then A has exactly one fixed point x^* in $\overset{\circ}{P}$. Moreover, constructing successively the sequence*

$$x_n = A(x_{n-1}, y_{n-1}), \qquad y_n = A(y_{n-1}, x_{n-1}), \quad n = 1, 2, \ldots, \tag{3.3}$$

for any initial $x_0, y_0 \in P$, one has

$$\|x_n - x^*\| \longrightarrow 0, \qquad \|y_n - x^*\| \longrightarrow 0, \quad n \longrightarrow \infty. \tag{3.4}$$

When $\tau(t) = t$, $\varphi(t) = t^{\alpha(t)}$, $0 < \alpha(t) < 1$, $0 < t < 1$, the conditions (H_1) and (H_2) are automatically satisfied. So, one has

Corollary 3.3. *Let P be a normal cone of a real Banach space $E, h > \theta$. $A : P_h \times P_h \longrightarrow P_h$ is a mixed monotone operator. In addition, suppose that for all $0 < t < 1$, there exists $0 < \alpha(t) < 1$ such that*

$$A\left(tx, \frac{1}{t}y\right) \geq t^{\alpha(t)} A(x, y), \quad \forall x, y \in P_h,\ 0 < t < 1, \tag{3.5}$$

then A has exactly one fixed point x^* in P_h. Moreover, constructing successively the sequence

$$x_n = A(x_{n-1}, y_{n-1}), \qquad y_n = A(y_{n-1}, x_{n-1}), \quad n = 1, 2, \ldots, \tag{3.6}$$

for any initial $x_0, y_0 \in P_h$, one has

$$\|x_n - x^*\| \longrightarrow 0, \qquad \|y_n - x^*\| \longrightarrow 0, \quad n \longrightarrow \infty. \tag{3.7}$$

Remark 3.4. Corollary 3.3 is the main result in [5]. So, our results generalized the result in [5].

When $\tau(t) = t$, $\varphi(t) = t^{\beta}$, $0 < t < 1$, the conditions (H_1) and (H_2) are automatically satisfied. So, we have the following.

Corollary 3.5. *Let P be normal solid cone of E, and let $A : \overset{\circ}{P} \times \overset{\circ}{P} \to \overset{\circ}{P}$ be a mixed monotone operator. In addition, suppose that there exists $0 < \beta < 1$ such that*

$$A\left(tx, \frac{1}{t}y\right) \geq t^{\beta} A(x, y), \quad \forall x, y \in \dot{P},\ 0 < t < 1, \tag{3.8}$$

then A has exactly one fixed point x^* in $\overset{\circ}{P}$. Moreover, constructing successively the sequence

$$x_n = A(x_{n-1}, y_{n-1}), \qquad y_n = A(y_{n-1}, x_{n-1}), \quad n = 1, 2, \ldots, \tag{3.9}$$

for any initial $x_0, y_0 \in \overset{\circ}{P}$, one has

$$\|x_n - x^*\| \longrightarrow 0, \qquad \|y_n - x^*\| \longrightarrow 0, \quad n \longrightarrow \infty. \tag{3.10}$$

Remark 3.6. Corollary 3.5 is the main result in [4]. So, our results generalized the result in [4].

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